Individual Contest

Section A.

In this section, there are 12 questions. Fill in the correct answer on the space provided at the end of each question. Each correct answer is worth 5 points.

1. Arrange the numbers 2^{847}, 3^{539}, 5^{363}, 7^{308} and 11^{242} from the largest to the smallest.

【***Solution***】

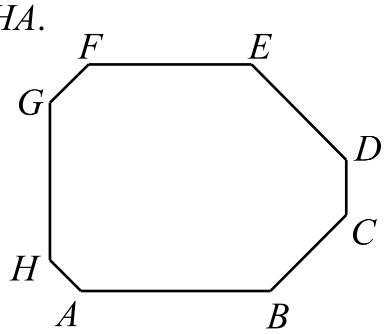
Since $7^4 = 2401 > 3^7 = 2187 > 2^{11} = 2048$, we have $7^{4\times77} > 3^{7\times77} > 2^{11\times77}$.

Since $2^7 = 128 > 5^3 = 125 > 11^2 = 121$, we have $2^{7\times121} > 5^{3\times121} > 11^{2\times121}$.

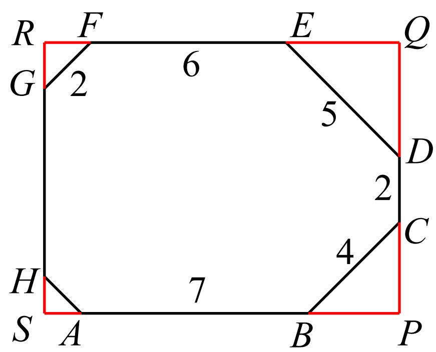
It follows that $7^{308} > 3^{539} > 2^{847} > 5^{363} > 11^{242}$

ANS: $7^{308} > 3^{539} > 2^{847} > 5^{363} > 11^{242}$

2. *ABCDEFGH* is an octagon in which all eight angles are equal. If *AB* = 7, *BC* = 4, *CD* = 2, *DE* = 5, *EF* = 6 and *FG* = 2, determine the sum of the lengths of *GH* and *HA*.



【***Solution***】



Extend *AB*, *CD*, *EF* and *GH* to form a quadrilateral *PQRS*. Each angle of *ABCDEFGH* is 135°. Hence each of *PCB*, *QED*, *RGF* and *SAH* are 45° −45° −90° triangles. It follows that *PQRS* is a rectangle. We have

$PB=PC=\dfrac{4}{\sqrt{2}}=2\sqrt{2}$, $QE=QD=\dfrac{5}{\sqrt{2}}=\dfrac{5\sqrt{2}}{2}$ and

$RG=RF=\dfrac{2}{\sqrt{2}}=\sqrt{2}$.

Hence

$SH = SA = SP - AB - BP = RQ - AB - BP = (\sqrt{2} + 6 + \dfrac{5\sqrt{2}}{2} - 7 - 2\sqrt{2}) = \dfrac{3\sqrt{2}}{2} - 1$ so that $HA = 3 - \sqrt{2}$. Also,

$$GH = RS - RG - SH = QP - RG - SH = (2\sqrt{2} + 2 + \frac{5\sqrt{2}}{2}) - \sqrt{2} - (\frac{3\sqrt{2}}{2} - 1) = 3 + 2\sqrt{2}$$

It follows that $GH + HA = 6 + \sqrt{2}$

Ans: $6+\sqrt{2}$

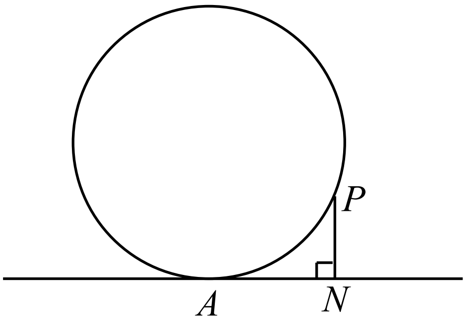
3. How many four-digit multiples of 9 are there if each of the digits are odd and distinct?

【***Solution***】

Since the sum of the four digits is also a multiple of 9, they must be 1, 3, 5 and 9. Any of the 4!=**24** permutations will yield a desired number.

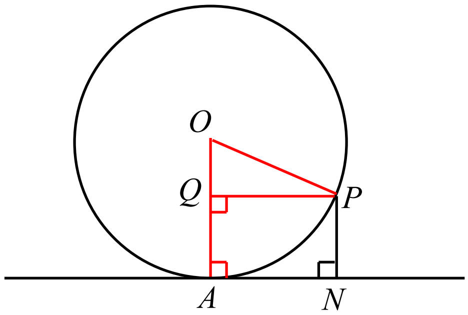
ANS: 24

4. A circle is tangent to a line at A. From a point P on the circle, a line is drawn such that PN is perpendicular to AN. If $PN = 9$ and $AN = 15$, determine the radius of the circle.



【***Solution***】

Let O be the center of the circle. Complete the rectangle $ANPQ$. Let r be the radius of the circle. Then



$$\begin{aligned} r^2 &= OP^2 \\ &= OQ^2 + QP^2 \\ &= (OA - PN)^2 + AN^2 \\ &= (r-9)^2 + 225 \end{aligned}$$

This simplifies to $18r = 306$ so that $r =$ **17**.

ANS: 17

5. From the first 30 positive integers, what is the maximum number of integers that can be chosen such that the product is a perfect square?

【***Solution 1***】

We first determine the prime factorization of the first 30 positive integers.

1	2	3	2×2	5	2×3
7	2×2×2	3×3	2×5	11	2×2×3
13	2×7	3×5	2×2×2×2	17	2×3×3
19	2×2×5	3×7	2×11	23	2×2×2×3
5×5	2×13	3×3×3	2×2×7	29	2×3×5

The product of all 30 numbers is $2^{26}\times3^{14}\times5^7\times7^4\times11^2\times13^2\times17\times19\times23\times29$. We must leave out 17, 19, 23 and 29, but this is still not the square of an integer until we also leave out 5. It follows that we can choose at most **25** numbers.

【***Solution 2***】

To count the number of 2's in the prime factorization of 30!, we collect a 2 from every second number, a second 2 from every fourth number, a third 2 from every eighth number, and a fourth 2 from every sixteenth number. The total is 15+7+3+1 =26. Similarly, the total number of 3's is given by $\left[\frac{30}{3}\right]+\left[\frac{30}{9}\right]+\left[\frac{30}{27}\right]=10+3+1$ $=14$, the number of 5's is $\left[\frac{30}{5}\right]+\left[\frac{30}{25}\right]=6+1=7$ and the number of 7's is $\left[\frac{30}{7}\right]=4$.

It is easy to see that the numbers of 11's and 13's are both 2, while the numbers of 17's, 19's, 23's and 29's are all 1. We can choose **25** numbers, all but 5, 17, 19, 23 and 29.

ANS: 25

6. Ace, Bea and Cec are each given a positive integer. They do not know the numbers given to the others, but are told that the sum of the three numbers is 15. Ace announces that he can deduce that the other two have different numbers, while Bea independently announces that she can deduce that no two of the three numbers are the same. Hearing both announcement, Cec announces that he knows all three numbers. What are they?

【***Solution***】

Let the numbers given to Ace, Bea and Cec be a, b and c respectively. Either all three are odd, or exactly one is odd. From the announcement of Ace, a must be even as otherwise Bea and Cec could have the same number. Similarly, b must also be even. Moreover, it has to be at least 8 as otherwise Bea may have the same number as either Ace or Cec. At this point, the possible combinations are (a, b, c)=(2, 8, 5), (4,8,3), (6,8,1), (2,10,3), (4,10,1) and (2,12,1). We cannot have c=1 or 3 since Cec will not be able to determine all three numbers. Hence c=**5**, b=**8** and a=**2**.

ANS: a=2, b=8, c=5

7. On the blackboard is a 3×3 magic square. The sum of the three numbers in each row, each column and each diagonal is the same. As shown in the diagram below, all but three of the numbers are erased. What is the number represented by x in the cell at the upper left corner?

x	21	94
3		

【***Solution***】

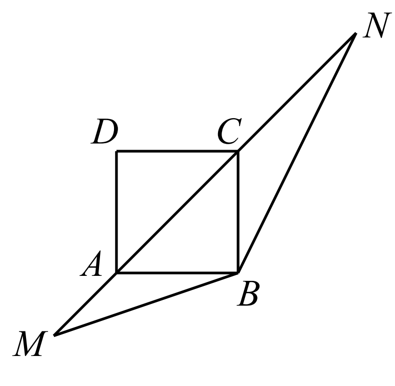
Label some of the other numbers as shown in the right diagram.

x	21	94
3	y	
z	a	b

Then 94+21+x=x+3+z so that z=112.
From 94+21+x=21+y+a, $a-x$=94−y.
From $x+y+b$=112+$a+b$, $a-x=y-112$.
Hence 94−y=y−112 so that y=103.
From 94+21+x=94+103+112, x=**194**.

ANS: 194

8. $ABCD$ is a square of side length 2009. M and N are points on the extension of the diagonal AC such that $\angle MBN$=135°. Determine the minimum length of MN.



【***Solution***】

Since

$\angle AMB = 45° - \angle ABM = 45° - (135° - 90° - \angle CBN) = \angle CBN$

.

Hence triangles *ABM* and *CBN* are similar, so that $\frac{2009}{CN}=\frac{AM}{2009}$.

Let *CN*=*x*, then

$$\begin{aligned} MN &= MA + AC + CN \\ &= \frac{2009^2}{x} + 2009\sqrt{2} + x \\ &= (\frac{2009}{\sqrt{x}} - \sqrt{x})^2 + 2\times 2009 + 2009\sqrt{2} \\ &\geq 2009\times(2+\sqrt{2}) \end{aligned}$$

ANS: $2009(2+\sqrt{2})$

9. Let x and y be positive integers such that $x\sqrt{y}+y\sqrt{x}-\sqrt{7x}-\sqrt{7y}+\sqrt{7xy}=7$. Determine $x+y$.

【***Solution*** **1**】

Let $a=\sqrt{x}$, $b=\sqrt{y}$ and $c=\sqrt{7}$. Then $a^2b+b^2a-ca-cb+abc=c^2$. Hence

$$\begin{aligned} 0 &= a^2b+b^2a-ca-cb+abc-c^2 \\ &= ab(a+b+c)-c(a+b+c) \\ &= (ab-c)(a+b+c) \end{aligned}$$

Since $a+b+c=\sqrt{x}+\sqrt{y}+\sqrt{7}>0$, we must have $ab-c=0$ or $\sqrt{xy}=\sqrt{7}$.

Since x and y are positive integers, (x, y)=(1, 7) or (7, 1). In either case, $x+y$=**8**.

【***Solution*** **2**】

We know $x\sqrt{y}+y\sqrt{x}+\sqrt{7xy}=7+\sqrt{7x}+\sqrt{7y}$.

That is, $\sqrt{xy}(\sqrt{y}+\sqrt{x}+\sqrt{7})=\sqrt{7}(\sqrt{7}+\sqrt{x}+\sqrt{y})$.

Hence $\sqrt{xy}=\sqrt{7}$, this imply $xy=7$.

Since x and y are positive integers, (x, y)=(1, 7) or (7, 1). In either case, $x+y$=**8**.

ANS: 8

10. There is a certain integer such that when we get its cube and its square, then each of the digits of the cube or square surprisingly contain only the numerals 1,2,3,4,5,6,7 and 8 exactly once in them. Determine this integer.

【***Solution***】

Since $20^2=400$ and $20^3=8000$ together use only 7 digits, our number is greater than 20. Since $32^2=1024$ and $32^3=32768$ together use 9 digits, our number is less than 32. The units digit of our number cannot be 0, 1, 5 or 6 as otherwise both the square and the cube end in the same digit. It cannot be 3 or 7 as otherwise the square ends in 9. Hence our number is 22, 24 or 28.

Since $22^2=484$, it is not 22. Since $28^3=21952$, it is not 28. Hence it must be **24**, and indeed $24^2=576$ and $24^3=13824$.

ANS: 24

11. We can express 2009 as the sum of four different numbers each of which consists of at least two digits and all the digits are identical, 2009=1111+777+88+33. What is the minimum number of addends needed to express 9002 in the same manner?

【***Solution***】

Let the sum of the four-digit numbers be 1111k, the sum of the three-digit numbers be 111m and the sum of the two-digit numbers be 11n. Each of k, m and n is at most 1+2+3+…+9=45. We have 9002=11(101k+10m+n)+m. Dividing by 11, we have

$$818 = 101k + 10m + n + \frac{m-4}{11}.$$

Now $\frac{m-4}{11}$ must be some non-negative integer q, so that m=11q+4.

Since $m \le 45$, we have $q \le 3$.

Now 818=101k+10(11q+4)+n+q, so that 778=101(k+q)+10q+ n.

Since $q \le 3$, 10q+$n \le 75$ so that we must have k+q=7.

Now 10q+n=71, so that q =3, k=4, n=41 and m=37.

We have n=41=1+2+3+…+9−4. We can either take out 4 or take out 1 and 3.

We choose the latter because we want to minimize the number of terms in the sum.

Similarly, m=37=1+2+3+…+9−8, and the best result is obtained by taking away 1, 2 and 5. It follows that 9002=4444+333+444+666+777+888+999+22+44+55+66+77+88+99, for a total of **14** terms.

ANS: 14

12. A farmer has ten baskets of eggs containing 12, 13, 14, 16, 18, 19, 22, 24, 29 and 34 eggs respectively. Some baskets have chicken eggs while other baskets have duck eggs. He sells one basket and found that the number of remaining chicken eggs is three times the number of the remaining duck eggs. How many eggs were in the basket he sold?

【***Solution***】

The total number of eggs in the nine baskets the farmer still has must be a multiple of 4. We look at the remainders when the numbers of eggs in the baskets is divided by 4.

Numbers	12	13	14	16	18	19	22	24	29	34
Remainders	0	1	2	0	2	3	2	0	1	2

The sum of the remainders is 1+2+2+3+2+3=13, so that when the total number of eggs is divided by 4, the remainder will be 1. Since the sum of the numbers of eggs in the unsold baskets is a multiple of 4, the number of eggs in the basket sold must leave a remainder of 1 when divided by 4. From the chart above, the only possibility is the basket with **13** eggs or the basket with **29** eggs. These in fact satisfied the remaining conditions since 3×(12+16+19)=14+18+22+24+29+34, or 3×(24+19)=12+13+14+16+18+22+34.

ANS: 13 or 29

Section B.

Answer the following 3 questions, and show your detailed solution on the space provided after each question. Each question is worth 20 points.

1. Each of the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 is to be placed in a different square of a 3×3 table. We color the largest number in each row, red while the smallest number in each row, green. Let M be the smallest among the three red numbers, and m be the largest among the three green numbers. Determine all possible values of $M - m$.

【***Solution***】

We have $3 \le M \le 7$, $3 \le m \le 7$ and $M \ne m$. (***5 points***)Hence the only possible values for $M-m$ are −4, −3, −2, −1, 1, 2, 3 and 4. (***5 points***)The diagrams below shows that each is possible. (***10 points***)

7	9	8
6	5	4
3	2	1

$M=3$，$m=7$

6	9	8
7	5	4
3	2	1

$M=3$，$m=6$

5	9	8
7	6	4
3	2	1

$M=3$，$m=5$

5	9	8
7	6	1
4	3	2

$M=4$，$m=5$

9	8	5
7	4	2
6	3	1

$M=6$，$m=5$

9	6	5
8	4	2
7	3	1

$M=7$，$m=5$

9	6	4
8	5	2
7	3	1

$M=7$，$m=4$

9	6	3
8	4	2
7	5	1

$M=7$，$m=3$

The respective values of $M-m$ are 3−7=**−4**, 3−6=**−3**, 3−5=**−2**, 4−5=**−1**, 6−5=**1**, 7−5=**2**, 7−4=**3** and 7−3=**4**.

ANS: −4, −3, −2, −1, 1, 2, 3 and 4

2. You are transporting mangoes by aircraft from Manila to Singapore. There are 12 planes available with the following weight capacities: 2, 2, 3, 3, 4, 7, 8, 8, 10, 10, 11 and 13 tons. Since no two planes may be assigned to the same route, then you may direct each plane to one of the following 12 routes:

Bangkok–Singapore	Hong Kong–Kuala Lumpur
Hong Kong–Singapore	Jakarta–Singapore
Kuala Lumpur–Bangkok	Kuala Lumpur–Singapore
Manila–Hong Kong	Manila–Jakarta
Manila–Kuala Lumpur	Manila–Taipei
Taipei–Bangkok	Taipei–Hong Kong

What is the maximum number of tons of mangoes you can ship from Manila to Singapore?

【***Solution***】

The chart below shows that it is possible to ship 35 tons (***5 points***) of mangoes from Manila to Singapore. (***5 points***)

Route	Capacity	Shipment
Bangkok–Singapore	4	4
Hong Kong–Kuala Lumpur	2	1
Hong Kong–Singapore	13	13
Jakarta–Singapore	8	8
Kuala Lumpur–Singapore	10	10
Kuala Lumpur–Bangkok	2	1
Manila–Kuala Lumpur	10	10
Manila–Taipei	7	6
Manila–Hong Kong	11	11
Manila–Jakarta	8	8
Taipei–Hong Kong	3	3
Taipei–Bangkok	3	3

It is not possible to ship more than 35 tons of mangoes from Manila to Singapore. Note that there are four routes leading from Manila and four other routes leading into Singapore. The highest capacities that can be assigned to these eight routes are 13, 11, 10, 10, 8, 8, 7 and 4, for a total of 71 tons. Hence either at most 35 tons of mangoes can come out of Manila or at most **35** tons of mangoes can get into Singapore. (***10 points***)

ANS: 35 tons

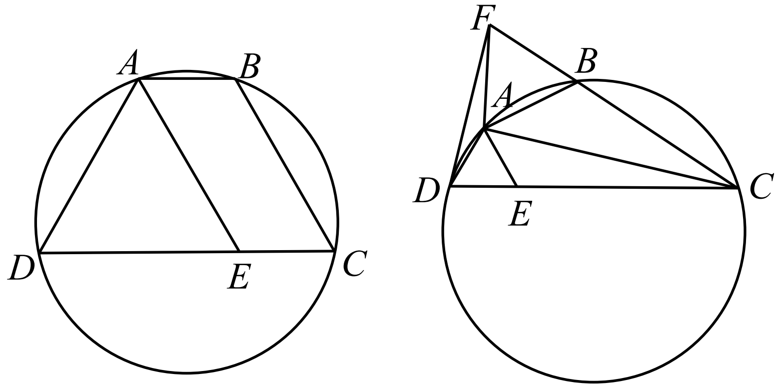
3. A, B, C and D are four consecutive points on a circle, such that $AB = 1$, $BC = 2$, $CD = 3$ and $\angle CDA = 60°$. Determine all possible lengths of DA.

【***Solution***】

Since $\angle CDA = 60°$, AC is the second longest side in triangle ACD. If $DA \geq 3$, then $AC \geq 3 = AB + BC$, which is a contradiction. Hence $DA < 3$. (***5 points***) Let E be the point on CD such that $DE = DA$. Then ADE is an equilateral triangle. We have

$$\angle AEC = 180° - \angle AED = 120° = 180° - \angle EDA = \angle ABC$$

Suppose $\angle BCD = 60°$. Then $ABCE$ is a parallelogram and $DA = EA = BC = 2$. (***5 points***)



If $\angle BCD \neq 60°$, extend CB to F such that $BF = 1$. Then BAF is an equilateral triangle, so that $\angle BFA = 60°$. Now $CF = CB + BF = 2 + 1 = 3 = CD$. Hence $\angle CFD = \angle CDF$, so that $\angle AFD = \angle CFD - 60° = \angle CDF - 60° = \angle ADF$. It follows that $DA = FA = AB = 1$. (***10 points***)

In summary, there are two possible lengths of DA, **2** or **1**. (***answer only 5 points***)

ANS: 2 or 1