Team Contest

1. Let x and y be real numbers such that $x^2+xy+y^2=3$. Find the smallest and largest values of $2x^2-5xy+2y^2$.

【*Solution*】

The given expression may be rewritten as $(x+y)^2=x^2+2xy+y^2=3+xy$ and $(x-y)^2=x^2-2xy+y^2=3-3xy$.

Since the square of a real number is non-negative, $3+xy\ge 0$ and $3-3xy\ge 0$, so that $-3\le xy\le 1$.

Now $2x^2-5xy+2y^2=2(x^2+xy+y^2)-7xy=6-7xy$. We have
$6-7xy\le 6-7(-3)=27$ and $6-7xy\ge 6-7(1)=-1$.

Hence $-1\le 2x^2-5xy+2y^2\le 27$. The lower bound is attained when x=1 and y=1, and the upper bound is attained when $x=\sqrt{3}$ and $y=-\sqrt{3}$.

Ans: The largest value is 27 and the smallest value is -1

2. There are n necklaces. In the first necklace, there are 5 beads, in the second necklace, there are 7 beads, and in the i-th necklace there are i beads more than the $(i-1)$st necklace for $i\ge 2$. Find the total number of beads in these n necklaces.

【*Solution*】

The total number of beads is given by

$$(4+1)+(4+1+2)+(4+1+2+3)+\cdots+(4+1+2+3+\cdots+n)$$
$$=4n+\frac{1}{2}(1\times1+1+2\times2+2+3\times3+3+\cdots+n^2+n)$$
$$=4n+\frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}$$
$$=\frac{48n+n(n+1)(2n+1)+3n(n+1)}{12}$$
$$=\frac{48n+2n^3+3n^2+n+3n^2+3n}{12}$$
$$=\frac{n(n^2+3n+26)}{6}$$

Ans: $\dfrac{n(n^2+3n+26)}{6}$

3. The positive integers x and y have 18 and 12 positive factors respectively. If their greatest common divisor is 24, find their least common multiple.

【*Solution*】

We have $18=6\times3=9\times2=3\times3\times2$. Hence x is of the form p^{17}, p^5q^2, p^8q or p^2q^2r. We have $12=4\times3=6\times2=3\times2\times2$. Hence y is of the form p^{11}, p^3q^2,

p^5q or p^2qr. Now $24 = 2^3 \times 3$. The power of 2 here is only matched if $y = 2^3 \times 3^2$. In *x*, the power of 3 must be 1 and the power of 2 must exceed 3. Hence $x = 2^8 \times 3$ and the least common multiple is $2^8 \times 3^2 = 2304$.

Ans: 2304

4. A metal wire of length 24 is to be bent into a triangle with integral side lengths. How many different such triangles are there?

【***Solution***】

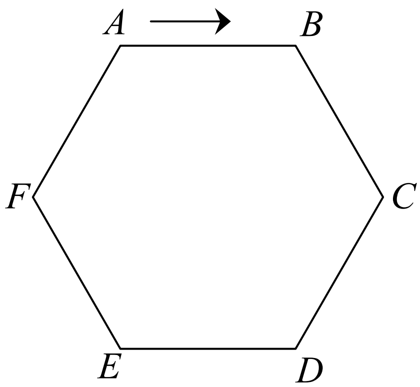
Let the side lengths be $a \le b \le c$, with $a+b+c=24$. Now $24-c=a+b>c$.

Hence $c \le 11$. On the other hand, $c \ge \dfrac{a+b+c}{3} = 8$.

For $c=8$, we can only have $(a, b, c)=(8, 8, 8)$.
For $c=9$, we can have (6, 9, 9) or (7, 8, 9).
For $c=10$, we can have (4, 10, 10), (5, 9, 10), (6, 8, 10) or (7, 7, 10).
For $c=11$, we can have (2, 11, 11), (3, 10, 11), (4, 9, 11), (5, 8, 11) or (6, 7, 11).
The total number is 1+2+4+5=12.

Ans: 12

5. An ant is at vertex *A* of a regular hexagon *ABCDEF* of unit side length, crawling along its perimeter. In the first move, it reaches vertex *B*. In each subsequent move, it crawls twice the distance of the preceding move. What is the total distance it has crawled after 2009 moves, and at which vertex will it be?

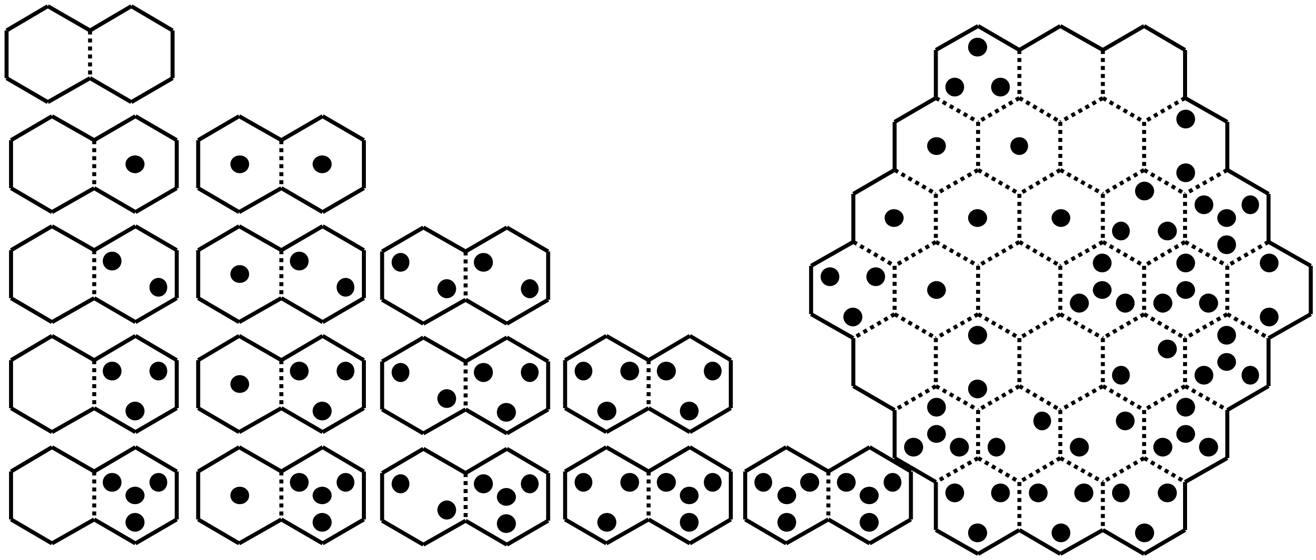


【***Solution***】

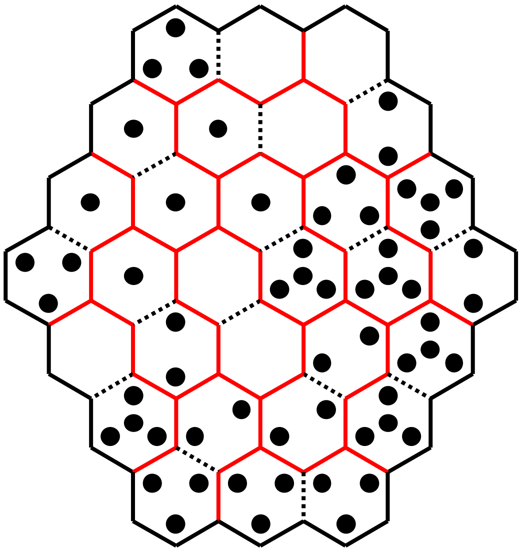
The total distance crawled by the ant is $T = 1 + 2 + 2^2 + \cdots + 2^{2008}$.
Then $2T = 2 + 2^2 + 2^3 + \cdots + 2^{2009}$, so that $T = 2^{2009} - 1$.
When successive powers of 2 are divided by 6, the remainders are 1, 2, 4, 2, 4 and so on, repeating in a cycle of length 2 after the zeroth power.
Hence the ant stops alternately at *B* and *D*. After 2009 moves, the ant will stop at *B*.

Ans: The total distance is $2^{2009} - 1$ and the ant stops at *B*

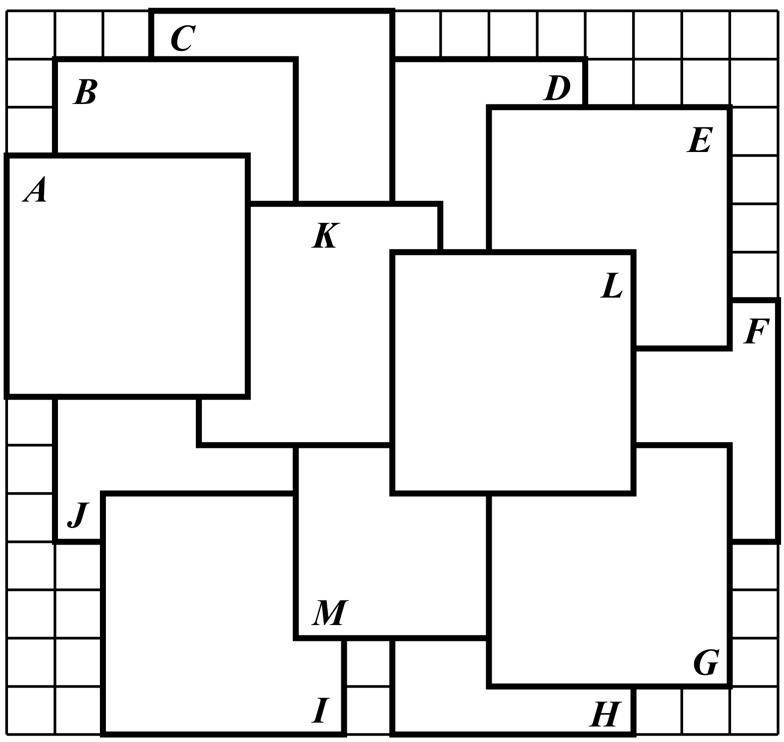
6. The fifteen hexagonal dominoes have been placed in a hex grid, but the borders of the pieces are not shown. Determine the proper placement of all the dominoes by drawing the borders. The dominoes may be rotated and flipped over, but not overlapped.



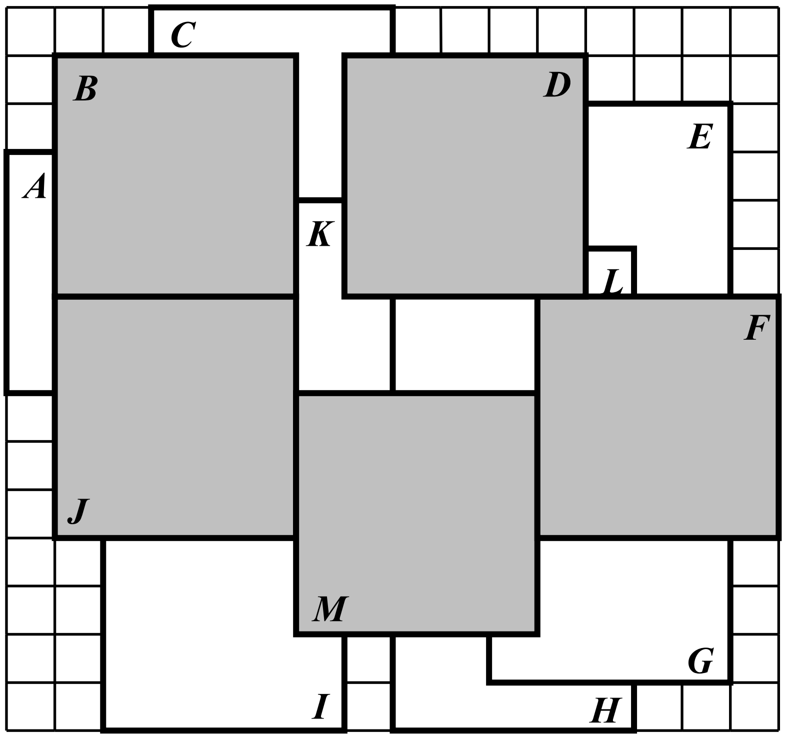
【*Solution*】



7. The 15×16 computer screen shows 13 overlapping 5×5 squares. Remove 8 squares so that the remaining 5 squares will not overlap, however the squares may touch one another along an edge.



【***Solution***】



Ans: Make *A*, *C*, *E*, *G*, *H*, *I*, *K* and *L* disappear

8. Leonardo of Pisa, son of Bonacci, was called Fibonacci. He is primarily known to us through a problem on reproductive rabbits. However, that problem only took up half a page of his monumental work *Liber Abaci*, or *The Book of Calculations*. What this book is to arithmetic is comparable to what Euclid -- *The Elements* is to geometry.
   The following problem is from *Liber Abaci* Chapter Thirteen, try to solve it.
   The first and second men said to the third and fourth, "If each of you gives us 1/3 of your bezants, then we have just enough money to buy that horse."
   The second and third men said to the fourth and fifth, "If each of you gives us 1/4 of your bezants, then we have just enough money to buy the same horse."
   The third and fourth men said to the fifth and first, "If each of you gives us 1/5 of your bezants, then we have just enough money to buy the same horse."
   The fourth and fifth men said to the first and second, "If each of you gives us 1/6 of your bezants, then we have just enough money to buy the same horse."
   The fifth and first men said to the second and third, "If each of you gives us 1/7 of your bezants, then we have just enough money to buy the same horse."
   How many bezants did each man have and how many bezants did the horse cost? (All the five men had a positive integral number of bezants; the cost of horse is also a positive integral number of bezants less than 5000.)

【***Solution***】

Let the cost of the horse is *H*, and the first, second, third, fourth and fifth men have X_1, X_2, X_3, X_4 and X_5, respectively. Thus we have

$$\begin{cases} H = X_1 + X_2 + \frac{1}{3}(X_3 + X_4) \\ H = X_2 + X_3 + \frac{1}{4}(X_4 + X_5) \\ H = X_3 + X_4 + \frac{1}{5}(X_5 + X_1) \\ H = X_4 + X_5 + \frac{1}{6}(X_1 + X_2) \\ H = X_5 + X_1 + \frac{1}{7}(X_2 + X_3) \end{cases} \Rightarrow \begin{cases} X_3 + X_4 = 3H - 3(X_1 + X_2) \\ X_4 + X_5 = 4H - 4(X_2 + X_3) \\ X_5 + X_1 = 5H - 5(X_3 + X_4) \\ X_1 + X_2 = 6H - 6(X_4 + X_5) \\ X_2 + X_3 = 7H - 7(X_5 + X_1) \end{cases}$$

Hence we get

$$\begin{aligned} X_1 + X_2 &= 6H - 6(X_4 + X_5) = 6H - 6(4H - 4(X_2 + X_3)) = -18H + 24(7H - 7(X_5 + X_1)) \\ &= 150H - 168(5H - 5(X_3 + X_4)) = -690H + 840(3H - 3(X_1 + X_2)) \\ &= 1830 - 2520(X_1 + X_2) \end{aligned}$$

So $2521(X_1 + X_2) = 1830H$. Since all of the five men had positive integer bezants and the cost of horse is positive integer bezants and less than 5000 bezants, X_1, X_2, X_3, X_4 and X_5 are positive integers and 2521 is a prime, hence H=2521 and $X_1 + X_2 = 1830$.

Therefore, $X_3 + X_4 = 2073$, $X_5 + X_1 = 2240$, $X_2 + X_3 = 1967$ and $X_4 + X_5 = 2216$.

Thus $X_1 + X_2 + X_3 + X_4 + X_5 = \frac{1}{2} \times (1830 + 2073 + 2240 + 1967 + 2216) = 5163$.

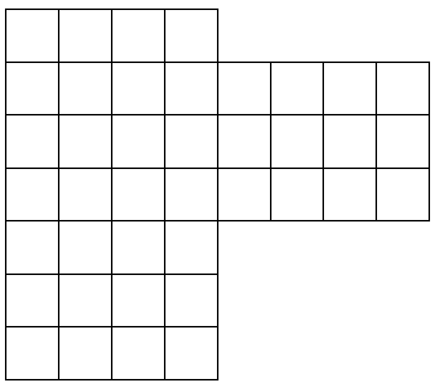
$$X_1 = 5163 - (X_2 + X_3) - (X_4 + X_5) = 980$$
$$X_2 = 5163 - (X_3 + X_4) - (X_5 + X_1) = 850$$

So

$$X_3 = 5163 - (X_1 + X_2) - (X_4 + X_5) = 1117$$
$$X_4 = 5163 - (X_2 + X_3) - (X_5 + X_1) = 956$$
$$X_5 = 5163 - (X_1 + X_2) - (X_3 + X_4) = 1260$$

Ans: The respective amounts were 980, 850, 1117, 956, 1260 and 2521 bezants.

9. Cut the following figure into two identical pieces. The pieces may be rotated, reflected or translated.

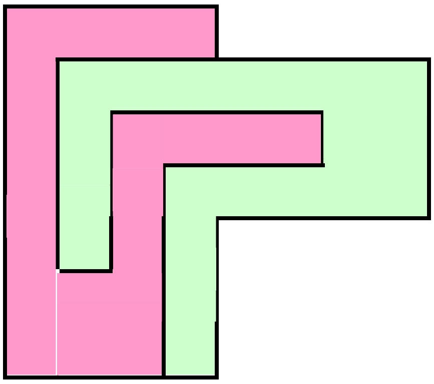


【***Solution***】

Call the two pieces A and B. We mark each square according to which piece it belongs. We start off marking the square at the bottom left corner with a boldfaced A and the square at the top right corner with a boldfaced B, to signify that they are corresponding squares in the two pieces. Since A can extend upward only to the seventh level, B can only extend leftward for six more columns, so that all squares in the first column on the left belongs to A. All squares in the second row from the top belong to B, except for the first one. All squares in the top row belong to

A, and the corresponding squares of B in the second column from the left are marked. Continuing this way, we obtain the dissection shown in the diagram below.

A	A	A	A				
A	B	B	B	B	B	B	**B**
A	B						
A	B						
A	B						
A							
A							



10. An off-shore oil-rig is pumping oil from the sea at the rate of 1 barrel per minute, and consumes water at the rate of 0.1 barrel per minute. A pipeline connecting the oil-rig to shore is used to pump oil to shore and water to the oil-rig. When the oil tap is turned on, it takes 6 minutes for the oil to reach shore, and when the oil tap is turned off, it takes 6 minutes before the pipeline is free of oil. When the water tap is turned on, it takes 6 minutes for the water to reach the oil-rig, and when the water tap is turned off, it takes 6 minutes before the pipeline is free of water. On the oil-rig, there is a large oil drum and a 13.2 barrel capacity water drum. What is the minimum rate per minute of transmission for the pipe-line?

【***Solution***】

A cycle consists of a period of time during which oil is pumped, 6 minutes for the oil to clear, another period of time during which water is pumped, and a final 6 minutes for the water to clear. We then return to the start of another cycle.

Let T minutes be the length of the cycle and t minutes be the amount of time pumping water, at rate r barrels per minute.

The total amount of water consumed is $\dfrac{T}{10}$ barrels and the total amount of water pumped is rt barrels. Hence $T = 10rt$.

There are $T - t$ minutes during which water is not being pumped. Hence the water drum must hold at least $0.1(T - t) = 13.2$ barrels.

The total amount of oil extracted is T barrels and the total amount of oil pumped is $r(T - t - 12)$ barrels. Hence $T = r(T - t - 12)$.

Substituting $T = 10rt$ into the other two equations, we have

$$10rt - t = 132 \text{ and } 11t = 10rt - 12.$$

Adding these yields $10t = 120$ so that $t = 12$. It follows that $T = 144$ and $r = 1.2$. The minimum rate of transmission of the pipe-line is 1.2 barrels per minute.

Ans: 1.2 barrels per minute