Individual Contest

1. Find the smallest positive integer whose product after multiplication by 543 ends in 2009.

【***Solution***】

It is obvious that the units digit of the required integer should be 3.
Assume that the tens digit of the required integer is a.

$$\begin{array}{cccccc} & & & & a & 3 \\ \times & & & 5 & 4 & 3 \\ \hline & & & ? & ? & 9 \\ & & ? & ? & 2 & \\ & & ? & 5 & & \\ \hline & ? & 2 & 0 & 0 & 9 \end{array}$$

Since the tens digit of the product is 0, the units digit of the sum of the units digits of 3×4=12 and 3a must be 0. So a=6.
Assume that the hundreds digit of the required integer is b.

$$\begin{array}{cccccc} & & & b & 6 & 3 \\ \times & & & 5 & 4 & 3 \\ \hline & & ? & ? & 8 & 9 \\ & ? & ? & 5 & 2 & \\ & ? & 1 & 5 & & \\ \hline & ? & 2 & 0 & 0 & 9 \end{array}$$

Since the hundreds digit of the product is 0, we know that the units digit of the sum of the units digits of 5×3=15, 4×6=24 and 3b, the tens digit of 4×3=12, 3×6=18 and 8+2=10 is 0. So b=6. However, 663×543=360009. So this is not the answer. Assume that the thousands digit of the required integer is c.

$$\begin{array}{cccccc} & & c & 6 & 6 & 3 \\ \times & & & 5 & 4 & 3 \\ \hline & ? & ? & 9 & 8 & 9 \\ ? & ? & 6 & 5 & 2 & \\ ? & 3 & 1 & 5 & & \\ \hline & ? & 2 & 0 & 0 & 9 \end{array}$$

Since the thousands digit of the product is 2, we know the units digit of the sum of the units digits of 5×6=30, 4×6=24 and 3c, the tens digits of 5×3=15, 4×6+1=25, 3×6+1=19 and 9+5+5+1=20 is 2. So c=4. Since 4663×543=2532009, **4663** is the answer.

ANS: 4663

2. Linda was delighted on her tenth birthday, 13 July 1991 (13/7/91) when she realized that the product of the day of the month together with the month in the year was equal to the year in the century: 13×7 = 91. She started thinking about other occasions in the century when such an event might occur, and imagine her

surprise when she realized that her two younger brothers would encounter a similar relationship on their tenth birthdays also. Given that the birthdays of the two boys are on consecutive days, when was Linda's youngest brother born?

【*Solution*】

The dates after Linda's tenth birthday when the special relationship holds are:

23/4/92	24/4/96	8/12/96
31/3/93	16/6/96	11/9/99
19/5/95	12/8/96	9/11/99

So the brothers' tenth birthdays must be on 23 April 1992, and 24 April 1996. Thus Linda's youngest brother was born on **24 April 1986**.

***ANS*:** 24 / 4 / 1986.

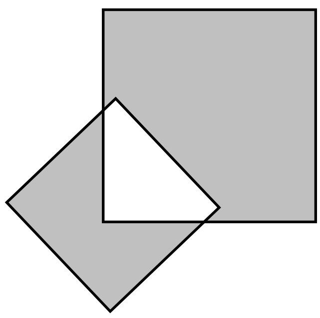
3. Philip arranged the number 1, 2, 3, ... , 11, 12 into six pairs so that the sum of the numbers in any pair is prime and no two of these primes are equal. Find the largest of these primes.

【*Solution*】

By observation, 11+12=**23** is the largest prime number. Let us explain why these six prime numbers must include 23:
From the given conditions, we know that the sum of these six prime numbers is equal to 1+2+…+11+12=78 and that each prime number does not exceed 23. So these prime numbers can only be six of 3, 5, 7, 11, 13, 17, 19, 23. If they are all less than 23, then the sum of the six prime numbers is at most 5+7+11+13+17+19=72＜78. So these six prime numbers must include 23.
Because 23=12+11, so the ten remaining numbers, with sum equal to 55, must be paired into five prime numbers. If they are all less than 19, then the sum of the five prime numbers can only be 5+7+11+13+17=53＜55. So these five numbers must include 19.
Because 19=10+9, so the eight remaining numbers, with sum equal to 36, must be paired into four prime numbers. Since 8+7=15＜17, these prime numbers can only be four of 3, 5, 7, 11, 13. Since 5+7+11+13=36, the four prime numbers can only be 5, 7, 11, 13. The matching can be done in one of the following two ways: 1+6=7, 2+3=5, 4+7=11, 5+8=13 or 1+4=5, 2+5=7, 3+8=11, 6+7=13.

***ANS*:** 23

4. In the figure, $\frac{3}{4}$ of the larger square is shaded and $\frac{5}{7}$ of the smaller square is shaded. What is the ratio of the shaded area of the larger square to the shaded area of the smaller square?



【*Solution*】

Assume that the area of the unshaded region is $2x$.

(i) The area of the larger square is $8x$ and hence the shaded area of the larger square is $6x$.

(ii) The area of the smaller square is $7x$ and hence the shaded area of the smaller square is $5x$.

So the ratio is 6x:5x=6:5.

ANS: 6:5

5. Observe the sequence 1, 1, 2, 3, 5, 8, 13, … . Starting from the third number, each number is the sum of the two previous numbers. What is the remainder when the 2009th number in this sequence is divided by 8?

【***Solution***】

Sequence	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
Reminder	1	1	2	3	5	0	5	5	2	7	1	0	1	1	2

The given sequence is a Fibonacci sequence. Notice the remainder form another series with mode 8 and cycle of 12.

2009=12×167+5, so the 2009th and the fifth remainder have the same value, 5. Hence the answer is **5**.

<u>ANS</u>: 5

6. Ampang Street has no more than 15 houses, numbered 1, 2, 3 and so on. Mrs. Lau lives in one of the houses, but not in the first house. The product of all the house numbers before Mrs. Lau's house, is the same as that of the house numbers after her house. How many houses are on Ampang Street?

【***Solution***】

Observe that, for positive integers not greater than 15, the numbers 7, 11 and 13 are prime numbers.

If Mrs. Lau lives in a house with number lower than 7, there are at most six houses on the street as no number smaller than 7 has the prime factor 7.

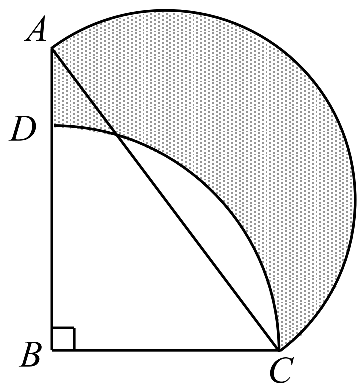
If Mrs. Lau lives in a house with number higher than 7, there are at least 14 houses on the street as 14 is the only other number in range that has 7 as a prime factor.

However, Mrs. Lau cannot live in both house number 11 and house number 13, and no other numbers in range have 11 or 13 as a prime factor.

It follows that Mrs. Lau must live in house number 7. Since the product of 1 to 6 is 20, which is equal to the product of 8, 9 and 10, there are **10** houses on the street.

<u>ANS</u>: 10

7. In the given figure, ABC is a right-angled triangle, where $\angle B = 90°$, $BC = 42$ cm and $AB = 56$ cm. A semicircle with AC as a diameter and a quarter-circle with BC as radius are drawn. Find the area of the shaded portion, in cm^2. (Use $\pi = \dfrac{22}{7}$)



【***Solution***】

According to Pythagoras theorem,

$AC^2 = AB^2 + BC^2 = 4900\,\text{cm}^2$.

Thus the area of the semicircle drawn on AC is

$$\frac{1}{2}\times\pi\times(\frac{1}{2}AC)^2 = \frac{1}{2}\times\frac{22}{7}\times 1225 = 1925\,\text{cm}^2.$$

The area of $\triangle ABC$ is

$$\frac{1}{2}\times AB\times BC=1176\,\text{cm}^2.$$

The area of quarter-circle *DBC* is

$$\frac{1}{4}\times\pi\times BC^2=\frac{1}{4}\times\frac{22}{7}\times 1764=1386\,\text{cm}^2.$$

Hence the area of the shaded portion is 1925+1176－1386=**1715** cm²

***ANS*:** 1715 cm^2

8. A number consists of three different digits. If the difference between the largest and the smallest numbers obtained by rearranging these three digits is equal to the original number, what is the original three-digit number?

【***Solution***】

Suppose the largest possible value of $\overline{xyz}$ after rearrangement is $\overline{abc}\,(a>b>c)$.

Then the smallest possible value after rearrangement is $\overline{cba}$.

As $\overline{xyz}=\overline{abc}-\overline{cba}=99(a-c)$, $\overline{xyz}$ is therefore a multiple of 99. There are 9 three-digit numbers which are multiples of 99, namely, 198, 297, 396, 495, 594, 693, 792, 891, 990. By testing, only **495** can satisfy the given conditions.

***ANS*:** 495

9. The last 3 digits of some perfect squares are the same and non-zero. What is the smallest possible value of such a perfect square?

【***Solution***】

The last digit of a perfect square can only be 0, 1, 4, 5, 6, 9. When its last digit is 1, 4, 5 or 9, its tens digit must be an even number. When its last digit is 6, its tens digit must be an odd number. The last three digits can only be 444. Since 444 is not a perfect square but $1444=38^2$. The smallest possible value of such a perfect square is 1444.

***ANS*:** 1444

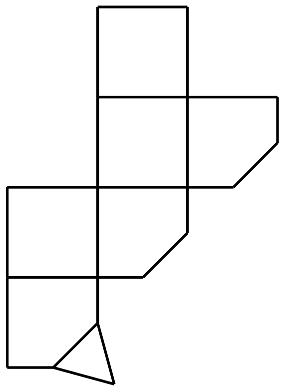
10. Lynn is walking from town *A* to town *B*, and Mike is riding a bike from town *B* to town *A* along the same road. They started out at the same time and met 1 hour after. When Mike reaches town *A*, he turns around immediately. Forty minutes after they first met, he catches up with Lynn, still on her way to town *B*. When Mike reaches town *B*, he turns around immediately. Find the ratio of the distances between their third meeting point and the towns *A* and *B*.

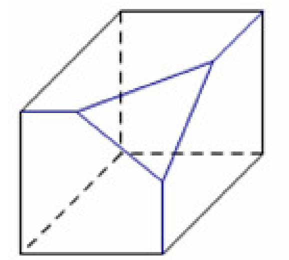
【***Solution***】

At the 3rd meeting, the total distance covered by Lynn and Mike equals 3 times the distance between A and B.Since it takes them 60 minutes to cover the distance once, they meet for the third time 180 minutes after they have started. Since the 1st meeting occurs after 60 minutes, and the 2nd meeting occurred 40 minutes after that, so between the 1st and 2nd meetings, the distance that Mike covers by riding, takes Lynn (60+60+40)=160 minutes to cover by walking. Hence the speed of Mike is 4 times the speed of Lynn. To walk from town A to town B, Lynn needs 5 hours. Before the third meeting, Lynn has walked for 3 hours, hence the ratio is **3:2**.

ANS: 3:2

11. The figure shows the net of a polyhedron. How many edges does this polyhedron have?





【***Solution* 1**】

By constructing the polyhedron as in the figure, we found that it has **15** edges.

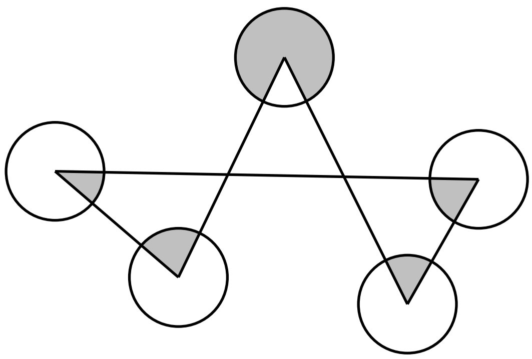
【***Solution* 2**】

Among all the segments of the net, if it is a common side of two faces, then it corresponds to an edge of the polyhedron, otherwise we require two segments to correspond to one edge of the polyhedron. Therefore the polyhedron has

$6+\frac{18}{2}=$**15** edges. ***ANS:*** 15

12. In the figure, the centers of the five circles, of same radius 1 cm, are the vertices of the triangles. What is the total area, in cm^2, of the shaded regions?

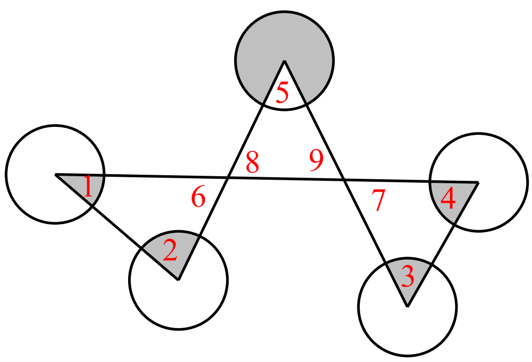
(Use $\pi=\frac{22}{7}$)



【***Solution***】

By labelling the angles in the figure, the total area of the shaded regions can be expressed as

$$\pi\times\frac{\angle 1+\angle 2+\angle 3+\angle 4+(360^\circ-\angle 5)}{360^\circ}$$



In the figure, $\angle 1+\angle 2+\angle 6=180^\circ$,

$\angle 3+\angle 4+\angle 7=180^\circ$, $\angle 5+\angle 8+\angle 9=180^\circ$,

and $\angle 6=\angle 8$, $\angle 7=\angle 9$.

Therefore

$$\angle 1+\angle 2+\angle 3+\angle 4+\angle 6+\angle 7=360^\circ$$
$$\angle 1+\angle 2+\angle 3+\angle 4+\angle 8+\angle 9=360^\circ$$
$$\angle 1+\angle 2+\angle 3+\angle 4+(180^\circ-\angle 5)=360^\circ$$
$$\angle 1+\angle 2+\angle 3+\angle 4+(360^\circ-\angle 5)=540^\circ$$

Hence the total area of the shaded regions is $\pi\times\frac{540^\circ}{360^\circ}=\frac{22}{7}\times\frac{3}{2}=\frac{33}{7}$ cm^2.

***ANS*:** $\frac{33}{7}$ cm^2

13. There are 10 steps from the ground level to the top of a platform. The 6th step is under repair and can only be crossed over but not stepped on. Michael walks up the steps with one or two steps only at a time. How many different ways can he use to walk up to the top of the platform?

【***Solution***】

Because the 6th step can only be crossed over but not stepped on and Michael walks up the steps with one or two steps only at a time, he must step on the 5th one and cross over the 6th to step on the 7th before moving up again. So the process of walking up to the top of the platform from ground level can be split into two stages: moving up 5 steps as the first stage and moving 3 steps as the second.

Let us use recurrence to find the number of ways in both stages:

When there is only 1 step, there is only 1 way of moving up.

When there are 2 steps, there are two ways of moving up: 1 step at a time or two steps at a time.

When there are 3 steps, if the first move covers only 1 step, then there are 2 ways of walking up the next two; if the first move covers 2 steps, then there is only 1 way of moving up the remaining step. So there are 2+1=3 ways of moving up 3 steps.

When there are 4 steps, if the first move covers only 1 step, then there are 3 ways of walking up the next three; if the first move covers 2 steps, then there are 2 ways of moving up the remaining 2 steps. So there are 3+2=5 ways of moving up 3 steps.

When there are 5 steps, if the first move covers only 1 step, then there are 5 ways of walking up the next four; if the first move covers 2 steps, then there are 3 ways of moving up the remaining 3 steps. So there are 5+3=8 ways of moving up 3 steps.

Therefore Michael can move up the steps in 8×3=**24** ways.

***ANS*:** 24

14. For four different positive integers *a*, *b*, *c* and *d*, where $a < b < c < d$, if the product $(d-c)\times(c-b)\times(b-a)$ is divisible by 2009, then we call this group of four integers a "friendly group". How many "friendly groups" are there from 1 to 60?

【***Solution***】

Note that $2009 = 7 \times 7 \times 41$. If (a, b, c, d) is a group of friendly numbers, then the difference between any two of them must be less than 60.

Since $(d - c) \times (c - b) \times (b - a)$ is divisible by 2009, one of the differences $d - c$, $c - b$ and $b - a$ must be equal to 41. Moreover, the two remaining differences must both be multiples of 7 (if one of them is a product of 49, then there will be a difference between two of the four numbers being not less than 49+41=90.)

If $d - c = 41$, then we can only have $c - b = 7$ and $b - a = 7$, otherwise one them will be at least equal to 14 and $d - a$ will be at least 41+14+7=62. In this case, $d - a = 55$. By choosing values for a, d and hence b and c are determined. We obtain thus 5 groups of friendly numbers: (a, b, c, d)=(1, 8, 15, 56), (2, 9, 16, 57), (3, 10, 17, 58), (4, 11, 18, 59) and (5, 12, 19, 60).

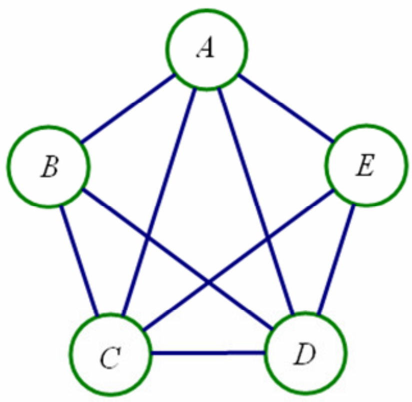
If $c - b = 41$, then, similarly, we can only have $d - c = 7$ and $b - a = 7$. There are also 5 groups of friendly numbers: (a, b, c, d) = (1, 8, 49, 56), (2, 9, 50, 57), (3, 10, 51, 58), (4, 11, 52, 59) and (5, 12, 53, 60).

If $b - a = 41$, then, similarly, we can only have $d - c = 7$ and $c - b = 7$. There are also 5 groups of friendly numbers: (a, b, c, d) = (1, 42, 49, 56), (2, 43, 50, 57), (3, 44, 51, 58), (4, 45, 52, 59) and (5, 46, 53, 60).

Therefore there are **15** groups of friendly numbers from 1 to 60.

ANS: 15

15. The figure shows five circles A, B, C, D and E. They are to be painted, each in one color. Two circles joined by a line segment must have different colors. If five colors are available, how many different ways of painting are there?



【***Solution***】

Because any two of the circles A, C, D and E are connected by a line segment, so any two of these four circles must be colored in different colors and thus four different colors are required to color them. So there are two different approaches to coloring the five circles:

(i) Use five colors to color the five circles. The number of ways is a permutation of the five colors, that is, $5 \times 4 \times 3 \times 2 \times 1$ =120 ways.

(ii) Use four colors to color the five circles with the colors of B and E being the same. This is equivalent to coloring the four circles A, B, C, D in four different colors. There are five ways of choosing four colors from five. Then we use the four chosen colors to color the four circles A, B, C, D. The number of ways of doing this is a permutation of four colors, that is, $4 \times 3 \times 2 \times 1 = 24$ ways.
So there are $5 \times 24 = 120$ ways of coloring based on this approach.
Altogether, there is a total of 120 + 120 = **240** ways of coloring.

ANS: 240