Team Contest

1. Below is a 3×60 table. Each row is filled with digits following its own particular sequence. For each column, a sum is obtained by adding the three digits in each column. How many times is the most frequent sum obtained?

Row A	1	2	3	4	5	1	2	3	4	5	…	4	5
Row B	1	2	3	4	1	2	3	4	1	2	…	3	4
Row C	1	2	1	2	1	2	1	2	1	2	…	1	2

【*Solution*】

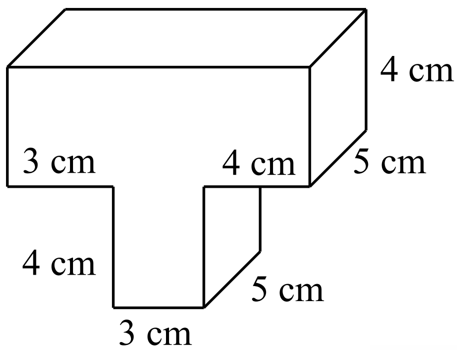
The sequence of each row is as below.

Row A	digits repeated in groups of 5
Row B	digits repeated in groups of 4
Row C	digits repeated in groups of 2

Since the L.C.M. of 5, 4 and 2 is 20, we can consider the sums of the first 20 columns, which are 3, 6, 7, 10, 7, 5, 6, 9, 6, 9, 5, 8, 5, 8, 9, 7, 4, 7, 8 and 11 respectively. Among them, 7 is the most frequent sum, appearing 4 times. Therefore the total number of times that 7 is obtained is：60÷20×4=**12**.

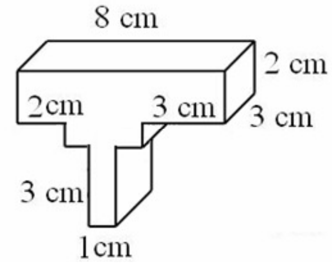
***ANS*:** 12

2. All surfaces of the T-shape block below is painted red. It is then cut into 1 cm×1 cm×1 cm cubes. Find the number of 1 cm×1 cm×1 cm cubes with all six faces unpainted.



【*Solution*】

On the right is the T-shape block that is left after removing the outer layer of painted surface.This T-shape block is unpainted.



The volume is：8×2×3+1×3×3+3×3×1=66 cm^3

Therefore, the number of required cubes is **66**.

***ANS*:** 66

3. Kiran and his younger brother Babu are walking on a beach with Babu walking in front. Each of Kiran's step measures 0.8 meter while each of Babu's step measures 0.6 meter. If both of them begin their walk along a straight line from the same starting point (where the first footprint is marked) and cover a 100 meter stretch, how many foot-prints are left along the path? (If a footprint is

imprinted on the 100 meter point, it should be counted. Consider two foot-prints as recognizable and distinct if one does not overlap exactly on top of the other.)

【***Solution***】

100 = 125×0.8, Number of Kiran's footprints = 126 (including the first one)

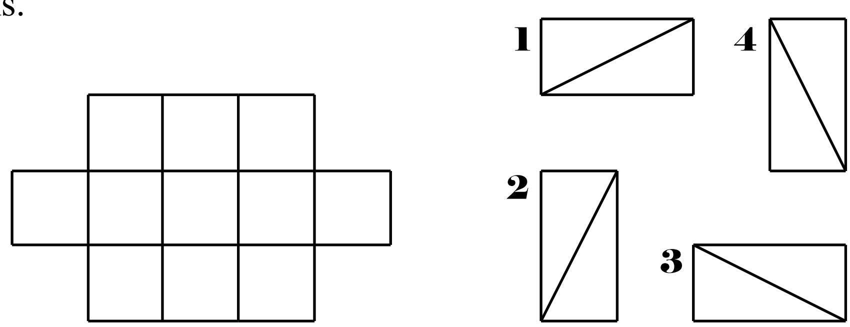
100=0.6x166+0.4, Number of Babu's footprints = 167 (including the first one)

100=2.4x41+1.6, Number of overlapped footprints = 42 (including the first one)

Visible footprints = 126 + 167 – 42 = **251**

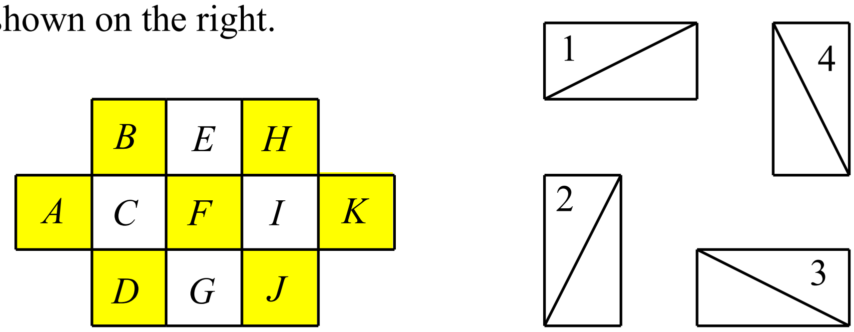
ANS: 251

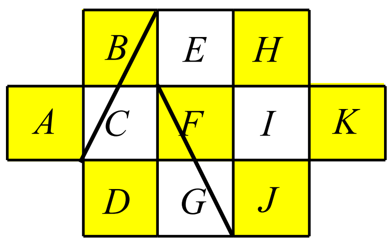
4. Four 2 ×1 cards, shown on the right in the following figure, are to be placed on the board shown on the left below, without overlapping and such that the marked diagonals of any two cards do not meet at a corner. The cards may not be rotated nor flipped over. Find all the ways of arranging these cards that satisfy the given conditions.



【***Solution***】

Color the squares in the chessboard alternately in yellow and white, and number the cards, as shown on the right.



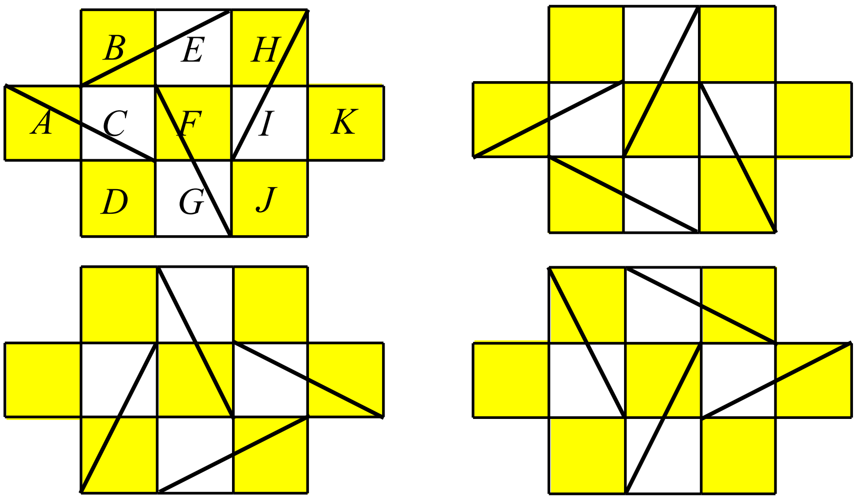


Because every card put on the chessboard covers one yellow and one white square. Putting all four cards on the chessboard will cover all the four white squares.

If card 2 is put on *BC*, then card 4 should not be put on *HI* or *IJ*, otherwise there will not be enough space to put cards 1 and 3. So card 4 can only be put on *FG*, as shown in the diagram. In this case, cards 1 and 3 can only be put on *EH* or *IK*. But neither card 1 nor card 3 can be put on *EH*. So there is no solution in this case.

If card 2 is put on *CD*, then card 4 should not be put on *HI* or *IJ*, otherwise there will not be enough space to put cards 1 and 3. So card 4 can only be put on EF. Using an argument similar to the previous one, there is no solution in this case.

If card 1 is put on *BE*, then card 4 should not be put on *CD* or *HI* or *IJ*, otherwise there will not be enough space to put cards 2 and 3. So card 4 can only be put on *FG*, as in the diagram below. In this case, card 2 can only be put on *HI* and card 3 can only be put on *AC*. So there is solution in this case. Because of symmetry of the chessboard and cards, we obtain other solutions after flipping and rotating, as in the following diagrams.



If card 1 is put on *CF*, then it will be impossible to put cards 2 and 4 simultaneously. So there is no solution in this case.
If card 1 is put on *AC*, then we can either put cards 2 and 4 on *FG* and *HI* respectively or on *EF* and *IJ* respectively. In the former case we cannot put card 3 and in the latter case we can put card 3 on *DG* and thus arrive at a solution, which has already been obtained earlier.
So there are only **4** ways of arranging these cards satisfying all the given conditions.

ANS: 4

5. Water is leaking out continuously from a large reservoir at a constant rate. To facilitate repair, the workers have to first drain-off the water in the reservoir with the help of water pumps. If 20 pumps are used, it takes 5 full hours to completely drain-off the water from the reservoir. If only 15 pumps are used, it will take an hour longer. If the workers are given 10 hours to complete the job of draining-off the water, what is the minimum number of water pumps required for the job?

【***Solution***】

Let the amount of water drained-off by 1 pump in 1 hour = 1 unit,

20 pumps × 5 hours = 100 unit, 15 pumps × 6 hours = 90 units,

Difference of 10 units in 1 hour = water lost through leaking.

Amount of water originally = $20\times5+10\times5=150$ units

Therefore water lost through leaking in 10 given hours = $10\times10 = 100$ units

(150 – 100) units ÷ 10 hours = **5** pumps

ANS: 5

6. As shown in the following figure, we arranged the positive integers into a triangular shape so that the numbers above or on the left must be less than the numbers below or on the right and each line has one more number than those above. Let us suppose a_{ij} stands for the number which is in the *i*-th line from the top and *j* is the count from the left in the triangular figure(e.g. a_{43}=9). If a_{ij} is 2009, what is the value of $i+j$?

1
2 3
4 5 6
7 8 9 10
…

【*Solution*】

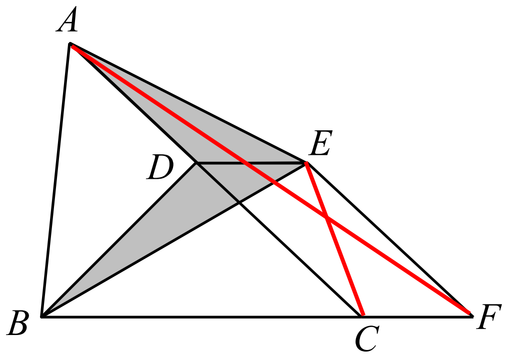
As shown in the figure, there are 1+2+…+n=$\frac{n(n+1)}{2}$ numbers in the first n lines.

So the last number of the i-th line is $\frac{i(i+1)}{2}$. The i-th line containing a_{ij}=2009 is the smallest positive integer satisfying the inequality $\frac{i(i+1)}{2} \geq 2009$. Because $\frac{62\times 63}{2}=1953$, $\frac{63\times 64}{2}=2016$, 1953＜2009＜2016, we get i=63. The first number in the 63-rd line is $\frac{62\times 63}{2}+1=1954$. So j=(2009－1954)+1=56, i+j=63+56=**119**.

ANS: 119

7. In the figure, the area of triangle ABC is 12 cm^2. $DCFE$ is a parallelogram with vertex D on the line segment AC and F is on the extension of line segment BC. If $BC=3CF$, find the area of the shaded region, in cm^2.

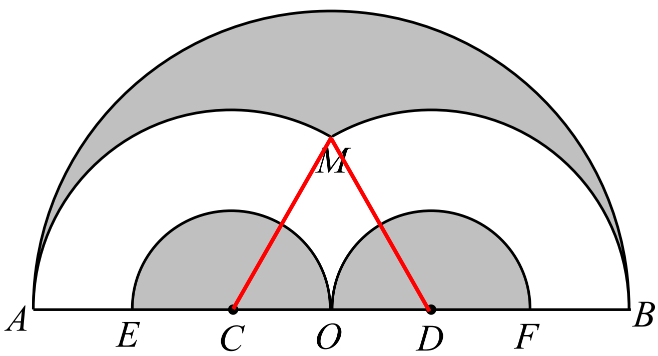
【*Solution*】



Because $DCFE$ is a parallelogram, therefore $DE//CF$ and $EF//CD$.
Join CE. Since $DE//CF$, that is, $DE//BF$. So $S_{\triangle DEB}=S_{\triangle DEC}$, and hence the area of the shaded region is equal to the area of triangle ACE.
Join AF. Since $EF//CD$, that is, $EF//AC$.
So $S_{\triangle ACE}=S_{\triangle ACF}$.
As $BC=3CF$ ，hence $S_{\triangle ABC}=3S_{\triangle ACF}$.
Therefore the area of the shade region is **4** cm^2.

<u>ANS:</u> 4 cm^2

8. In the figure below, the diameter AB of semi-circle O is 12 cm long. Points C and D trisect line segment AB. An arc centered at C and with CA as radius meets another arc centered at D and with DB as radius at point M. Take the distance from point M to AB as 3.464 cm. Using C as center and CO as radius, a semi-circle is constructed to meet AB at point E. Using D as center and DO as radius, another semi-circle is constructed to meet AB at point F.



Find the area of the shaded region. (Use $\pi = 3.14$ and give your answer correct to 3 decimal places.)

【*Solution*】

Join *CM* and *DM*. The given conditions imply that *CD*=*CM*=*DM*. Therefore triangle *CDM* is an equilateral triangle with side 4.

Therefore $\angle ACM = \angle BDM = 120^\circ$.

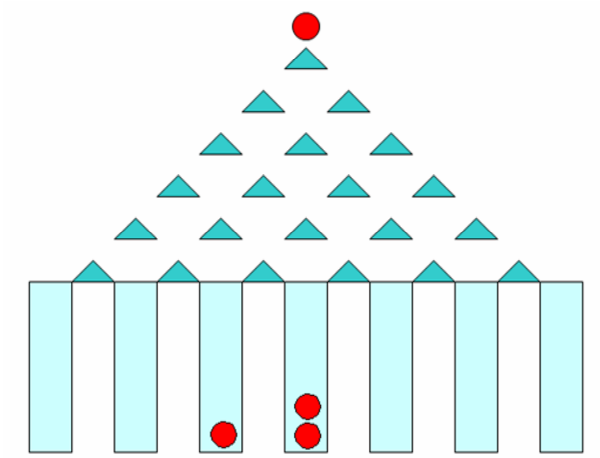
Since the radii of semi-circles *C* and *D* are both 2, and the distance from point *M* to *AB* is 3.464 cm, the area of equilateral triangle *CDM* is

$\frac{1}{2} \times 4 \times 3.464 = 6.928\,\text{cm}^2$. Hence the area of the shaded region is

$$3.14 \times 6^2 \times \frac{1}{2} - 3.14 \times 4^2 \times \frac{120}{360} \times 2 - 6.928 + 3.14 \times 2^2 = 28.659\,\text{cm}^2$$

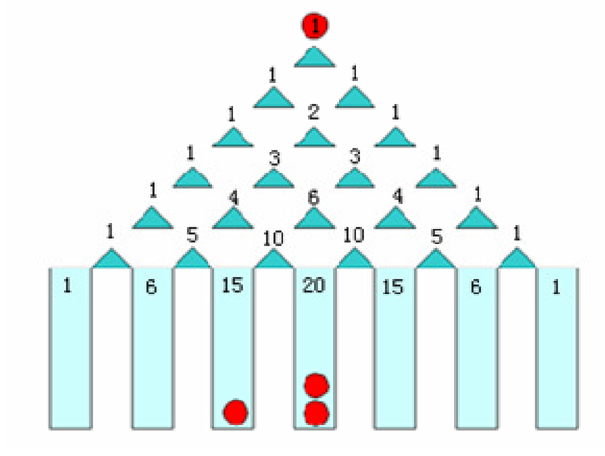
ANS: 28.659 cm^2

9. The following figure shows a famous model, designed by Galton, a British biostatistician, to test the stability of frequency. Some wooden blocks with cross-sections in the shape of isosceles triangles are affixed to a wooden board. There are 7 bottles below the board and a small ball on the top of highest block. As the small ball falls down, it hits the top vertices of some wooden blocks below and rolls down the left or right side of a block with the same chance, until it falls into a bottle. How many different paths are there for the small ball to fall from the top of the highest block to a bottle?



【*Solution*】

As shown in the following figure, the top vertex of every isosceles triangle and the mouth of each bottle are labeled with numbers which represent the number of different paths from the starting point to that position, e.g., there are two paths for the small ball to fall from the starting point to the vertex of the isosceles triangle labeled ‘2’ : the first path corresponds to the small ball hitting the top vertex of the first isosceles triangle in the second row

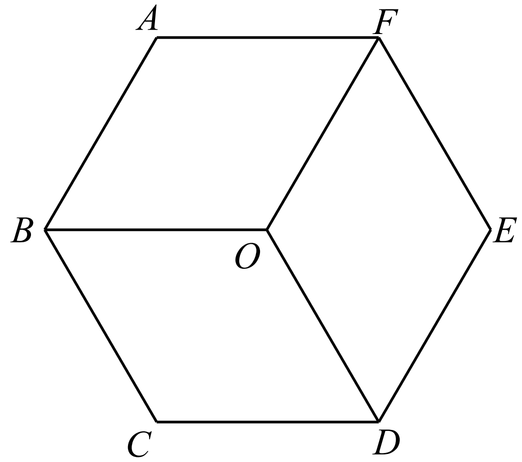


and then rolling down its right side; the second path corresponds to the small ball hitting the top vertex of the second isosceles triangle in the second row and then rolling down its left side.

So there are altogether 1+6+15+20+15+6+1=**64** different paths for the small ball to fall from the starting point into a bottle.

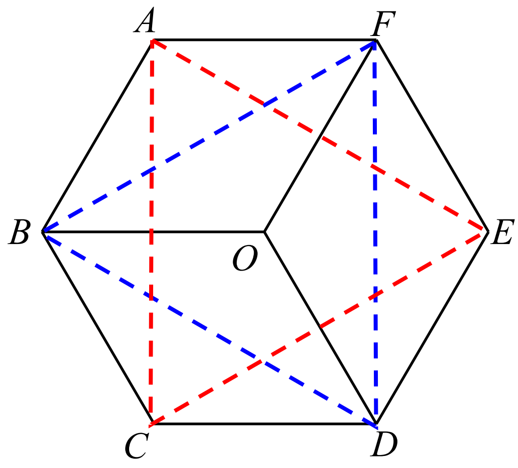
ANS: 64

10. In the following figure, assign each of the numbers 1, 2, 3, 4, 5, 6, 7 to one of the six vertices of the regular hexagon $ABCDEF$ and its center O so that sums of the numbers at the vertices of the rhombuses $ABOF$, $BCDO$ and $DEFO$ are equal. If solutions obtained by flipping or rotating the hexagon are regarded as identical, how many different solutions are there?



【***Solution***】

Because $A+B+O+F=B+C+D+O=D+E+F+O$, therefore $A+F=C+D$, $B+C=E+F$, $A+B=D+E$, or $C-A=F-D$, $E-C=B-F$, $E-A=B-D$. Hence the differences between the numbers at adjacent vertices of triangle ACE are the same as the differences between the numbers at adjacent vertices of triangle DFB.



When we are given three numbers to assign to the vertices of triangle ACE, if we regard solutions obtained by rotating or flipping the hexagon as the same, then there is only way of assigning the numbers.

Let us assume that $A < C < E$.

Let us first assign a number to the center O, and then divide the remaining six numbers into two groups with three numbers each to be assigned to the two groups of vertices $\{A, C, E\}$ and $\{B, D, F\}$.

From the above, once we have assigned the numbers to the group of vertices $\{A, C, E\}$, then the numbers assigned to the group of vertices $\{B, D, F\}$ are also determined.

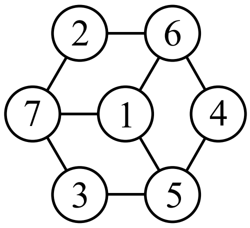
Let us discuss the different cases corresponding to assigning different numbers to the center O:

(1) If $O=1$, then we divide 2, 3, 4, 5, 6, 7into two groups and arrange the three numbers in each group in ascending order. Because $C-A=F-D$, $E-C=B-F$, $E-A=B-D$, so the difference between any two corresponding adjacent numbers in the two groups must be equal. If 2 and 3 are in the same group, then it can be easily verified that only the grouping $\{2, 3, 4\}$ with

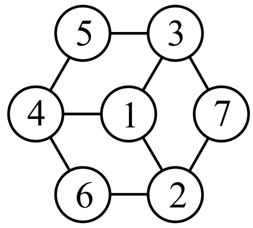
{5, 6, 7} satisfies the given conditions. If 2 and 4 are in the same group, then it can be easily verified that only the grouping {2, 4, 6} with {3, 5, 7} satisfies the given conditions. If 2 and 5 are in the same group, then 3 and 4 will be in another group but $5-2\neq4-3$, not satisfying the given conditions. Similarly, we can verify that there are no other ways of grouping satisfying the given conditions.

So there are 4 ways of grouping in this case:

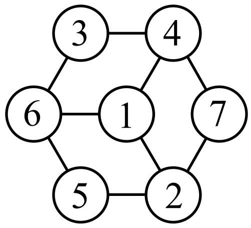
{A, C, E}={2, 3, 4}，
{D, F, B}={5, 6, 7}；



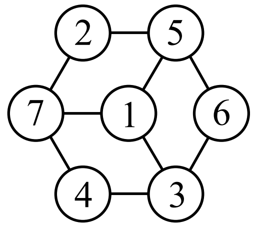
{A, C, E}={5, 6, 7}，
{D, F, B}={2, 3, 4}；



{A, C, E}={3, 5, 7}，
{D, F, B}={2, 4, 6}；

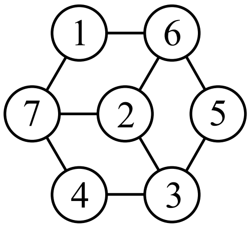


{A, C, E}={2, 4, 6}，
{D, F, B}={3, 5, 7}；

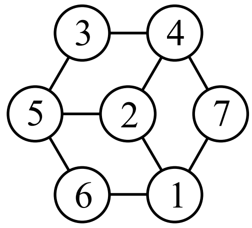


(2) If $O=2$, we obtain 2 ways of grouping using an analysis similar to (1):

{A, C, E}={1, 4, 5}，
{D, F, B}={3, 6, 7}；

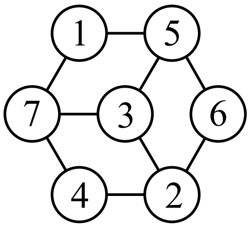


{A, C, E}={3, 6, 7}，
{D, F, B}={1, 4, 5}；

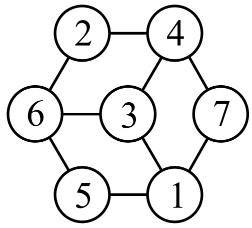


(3) If $O=3$, we obtain 2 ways of grouping using an analysis similar to (1):

{A, C, E}={1, 4, 6}，
{D, F, B}={2, 5, 7}；



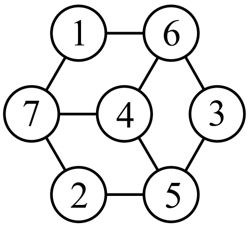
{A, C, E}={2, 5, 7}，
{D, F, B}={1, 4, 6}；

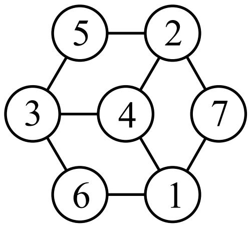


(4) If $O=4$, we obtain 2 ways of grouping using an analysis similar to (1):

$\{A, C, E\}=\{1, 2, 3\}$， $\{D, F, B\}=\{5, 6, 7\}$；

$\{A, C, E\}=\{5, 6, 7\}$， $\{D, F, B\}=\{1, 2, 3\}$；

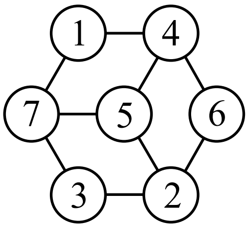


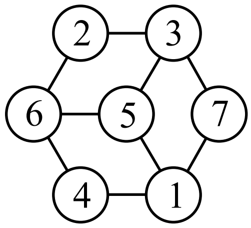


(5) If $O=5$, since 1, 2, 3, 4, 5, 6, 7 are seven consecutive integers, the situation is the same as that for $O=3$. We can easily obtain 2 ways of grouping:

$\{A, C, E\}=\{1, 3, 6\}$， $\{D, F, B\}=\{2, 4, 7\}$；

$\{A, C, E\}=\{2, 4, 7\}$， $\{D, F, B\}=\{1, 3, 6\}$；

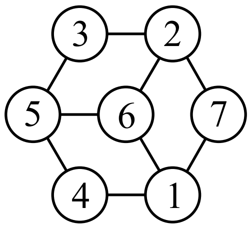


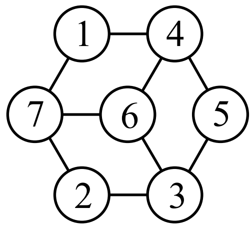


(6) If $O=6$, since 1, 2, 3, 4, 5, 6, 7 are seven consecutive integers，the situation is the same as that for $O=2$. We can easily obtain 2 ways of grouping:

$\{A, C, E\}=\{3, 4, 7\}$， $\{D, F, B\}=\{1, 2, 5\}$；

$\{A, C, E\}=\{1, 2, 5\}$， $\{D, F, B\}=\{3, 4, 7\}$；

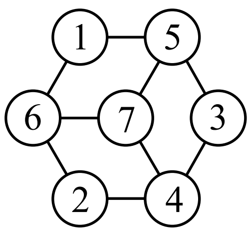


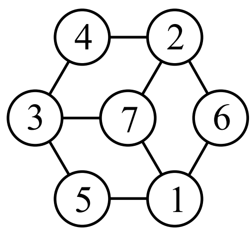


(7) If $O=7$, since 1, 2, 3, 4, 5, 6, 7 are seven consecutive integers， the situation is the same as that for $O=1$. There are 4 ways of grouping:

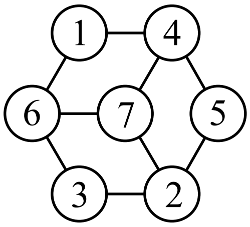
$\{A, C, E\}=\{1, 2, 3\}$， $\{D, F, B\}=\{4, 5, 6\}$；

$\{A, C, E\}=\{4, 5, 6\}$， $\{D, F, B\}=\{1, 2, 3\}$；

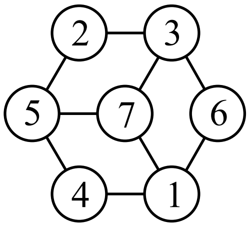




$\{A, C, E\}=\{1, 3, 5\}$, $\{D, F, B\}=\{2, 4, 6\}$;



$\{A, C, E\}=\{2, 4, 6\}$, $\{D, F, B\}=\{1, 3, 5\}$;



Altogether, there are 4+2+2+2+2+2+4=**18** ways of grouping. ANS: 18